Aerodynamic Coefficients of a Thin Elliptic Wing in Unsteady Motion

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An analytic solution is presented for the linearized lifting surface problem of a thin wing with an elliptic planform in unsteady incompressible flow. The analysis is based on the expansion of the acceleration potential in an infinite series of ellipsoidal harmonics and extends the steady analysis, recently developed by the authors, to the unsteady flow regime. Explicit expressions are obtained for both the starting lift in the case of impulsive acceleration and for the lift due to constant acceleration. The exact solution thus obtained is valid for the whole range of aspect ratios. The analytic result for the starting lift may thus be regarded as a new generalization of the classical Wagner's two-dimensional solution for planforms of finite aspect ratio.

Introduction

THIS paper is concerned with the analytic evaluation of lift and moment acting on a thin wing with an elliptic planform in linearized unsteady, incompressible flow. The solution concentrates on the uncambered wing and on two cases of variable forward velocity, namely, an impulsive acceleration to a constant speed and a constant acceleration. The analysis is based on the steady-state solution, recently presented by the authors,[1] in which closed-form expressions have been obtained for the steady lift and moment in terms of the arbitrary aspect ratio.

For the unsteady lifting surface problem, the existing analytic solutions are even scarcer then for the steady one. In fact, complete solutions are known only for the limits of the two-dimensional airfoil and the slender wing. Wagner[2] investigated the impulsively started motion of the two-dimensional airfoil and obtained the classical result that the starting lift slope coefficient is π, that is half of the steady-state value. More general theory for the two-dimensional wing has been developed by von Kármán and Sears.[3] Jones[4] suggested an approximate three-dimensional correction to the so-called Wagner function, which is valid only for an elliptic wing of large aspect ratio. For the other limit, i.e., for the small aspect ratio, Ando[5] evaluated the lift of a slender wing due to constant acceleration. The only attempts to analytically treat a truly three-dimensional unsteady lifting surface were due to Krienes and Schade[6] and Kochin.[7] However, their extremely complicated solutions apply only to a circular planform in a simple harmonic out-of-plane motion. To the authors' knowledge, Krienes'[8] steady analysis of a general elliptic planform (which is approximate in the sense that it requires inversion of a different infinite set of linear equations for each aspect ratio) has not been extended to the unsteady flow regime.

In the present solution, explicit analytic expressions are obtained for both the starting lift in the case of impulsive acceleration and the lift due to constant acceleration. The exact solution thus obtained is valid for the whole range of aspect ratios, from the two-dimensional limit through the wing of circular planform to the slender elliptic wing. The analysis is based on expansion of the acceleration potential in a series of ellipsoidal harmonics, using an ellipsoidal coordinate system in which the wing is represented by the fundamental elliptic disk.

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Formulation of the Boundary Value Problem

Let the center of the wing coincide with the origin of a Cartesian coordinate system, in which the z axis points upwards and the x axis is parallel and opposite to the direction of the wing's unsteady translatory velocity $U(t)$. The thickness, camber, twist, and angle of attack are considered small so that the requirements of linearized theory are satisfied and the boundary conditions may be applied on the projection of the wing on the xy plane. We formulate the problem is terms of the acceleration potential $\psi=[p(X,t)-p_\infty]/\rho U(t)$, where $X=(x,y,z)$, ρ is fluid density, p the pressure at a field point, and p_∞ the pressure at infinity. In the linear theory, ψ is harmonic function of X for all t, i.e.,

$$\nabla^2\psi(X,t)=0 \tag{1}$$

The vertical component of the disturbance velocity w ("downwash") is related to ψ by the linearized Euler equation of motion,

$$\frac{\mathrm{D}w}{\mathrm{D}t}(X,t)=-U(t)\frac{\partial\psi}{\partial z}(X,t) \tag{2}$$

where $\mathrm{D}/\mathrm{D}t=\partial/\partial t+U(t)\partial/\partial x$ is the substantial derivative. For a wing surface given by $z=z^*(x,y)$, the downwash is $w^*(x,y,t)=(\mathrm{D}/\mathrm{D}t)z^*(x,y)$, and, therefore, the kinematic condition on the wing is

$$U(t)\frac{\partial\psi}{\partial z}=-\frac{\mathrm{D}w^*}{\mathrm{D}t} \tag{3}$$

which is to be evaluated on the wing projection S on $z=0^\pm$.

The lifting problem thus formulated requires that ψ be an odd function of z, hence,

$$\psi(x,y,0,t)=0 \tag{4}$$

outside S.

The Kutta condition implies that

$$\psi(x_t,y_t,0,t)=0 \tag{5}$$

where the subscript t denotes the trailing edge.

Lifting surface theory suggests a square-root singularity of ψ along the leading edge, since the flow there behaves locally like the flow near the edge of a two-dimensional flat-plate airfoil (and, as a matter of fact, like the flow near the edge of a nonlifting disk in transverse flow). Hence, when $x \to x_l$ and $y \to y_l$,

$$\psi(x,y,0,t) \sim \epsilon^{-\frac{1}{2}} \tag{6}$$

where l denotes the leading edge and $\epsilon = [(x-x_l)^2 + (y-y_l)^2]^{\frac{1}{2}}$ is the normal distance measured from the leading edge.

Finally, since the disturbance pressure must vanish at infinity,

$$\psi(X,t) \to 0, \qquad |X| \to \infty \tag{7}$$

Equations (1) and (3-7) comprise the linear boundary value problem to be solved in the sequel.

It is convenient to introduce the distance τ traveled by the wing during time t,

$$\tau(t) = \int_0^t U(t')\mathrm{d}t' \tag{8}$$

assuming that $U = U[t(\tau)]$ is a one-valued function of τ. Regarding ψ and w as functions of X and τ, Eq. (2) becomes

$$\frac{\mathrm{D}w}{\mathrm{D}\tau}(X,\tau) = -\frac{\mathrm{D}\psi}{\mathrm{D}z}(X,\tau) \tag{9}$$

where $\mathrm{D}/\mathrm{D}\tau = \partial/\partial\tau + \partial/\partial x$ and $\partial/\partial t = U\partial/\partial\tau$. Hence, Eq. (3) becomes

$$-\frac{\partial\psi}{\partial z} = \frac{\mathrm{D}w^*}{\mathrm{D}\tau} = \frac{\mathrm{D}^2 z^*}{\mathrm{D}\tau^2} \tag{10}$$

on the wing projection S.

Applying the Laplace transform

$$\hat{f}(X,\tau) = \int_0^\infty e^{-s\tau} f(X,\tau)\mathrm{d}\tau$$

to Eqs. (9) and (10) under zero initial conditions yields

$$\left(s + \frac{\partial}{\partial x}\right)\hat{w}(X,s) = -\frac{\partial\hat{\psi}}{\partial z}(X,s) \tag{11}$$

and therefore

$$\left(s + \frac{\partial}{\partial x}\right)\hat{w}^*(x,y,s) = -\frac{\partial\hat{\psi}}{\partial z}(x,y,0,s) \tag{12}$$

Solving Eq. (11) for $\hat{w}$, using Eq. (7), yields

$$\hat{w}(X,s) = -e^{-sx}\int_{-\infty}^{x} e^{sx'}\frac{\partial\hat{\psi}}{\partial z}(x',y,z,s)\mathrm{d}x' \tag{13}$$

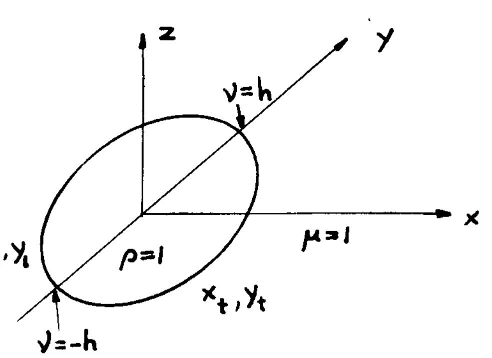

Fig. 1 Elliptic wing as a degenerate ellipsoid.

whereas on the wing projection,

$$w^*(x,y,z) = -e^{-sx}\int_{-\infty}^{x} e^{sx'}\frac{\partial\psi}{\partial z}(x',y,0,s)\mathrm{d}x' \tag{14}$$

To avoid difficulties in crossing the leading edge where ψ is singular, the integration in Eq. (14) should be considered as taken outside the plane and then evaluated in the limit $z \to 0^+$.

Now we introduce an ellipsoidal coordinate system (ρ,μ,ν) related to the Cartesian system by

$$x^2 = h^{-2}(1-h^2)^{-2}(\rho^2-h^2)(\mu^2-h^2)(h^2-\nu^2)$$
$$y^2 = h^{-2}(1-h^2)^{-1}\rho^2\mu^2\nu^2$$
$$z^2 = (1-h^2)^{-2}(\rho^2-1)(1-\mu^2)(1-\nu^2) \tag{15}$$

where $1 \le \rho \le \infty$, $h \le \mu \le 1$, $-h \le \nu \le h$. The surfaces $\rho = \text{const}$ are triaxial ellipsoids whose largest axes lie on the y plane. The wing projection is situated on the elliptic disk $\rho = 1$, that is, $x^2 + (1-h^2)y^2 \le 1$, $z = 0$. (See Fig. 1.) Hence, the minor axis of the ellipse (maximum chord) is taken as 2, the major axis is $2(1-h^2)^{-\frac{1}{2}}$, and h is the eccentricity. For the upper $(z = 0^+)$ and lower $(z = 0^-)$ sides of the disk, the square root of $(1-\mu^2)$ is taken as positive and negative, respectively. The plane $z = 0$ outside the disk is denoted by $\mu = 1$, with ρ varying between 1 and ∞, so that on the contour $\rho = \mu = 1$.

An external ellipsoidal harmonic which vanishes at infinity is $F_n^m(\rho)E_n^m(\mu)E_n^m(\nu)$, for $n = 0,1,\ldots$ and $m = 0, 1,\ldots,2n+1$, where E and F are the Lamé functions of the first and second kind, respectively. For the lifting problem, only harmonics that are odd with respect to z have to be introduced, namely (using Hobson's[9] notation) those corresponding to the classes $E = M$ and $E = N$, where $M(\lambda) = |\lambda^2 - 1|^{\frac{1}{2}}\bar{M}(\lambda)$, $N(\lambda) = |\lambda^2 - 1|^{\frac{1}{2}}\,|\lambda^2 - h^2|^{\frac{1}{2}}\bar{N}(\lambda)$, and $\bar{M}$ and $\bar{N}$ are simple polynomials of λ. These harmonics vanish along the whole edge of the disk and, therefore, cannot satisfy Eq. (6). On the other hand, the solution for the nonlifting transverse steady flow about an elliptic disk is

$$\psi = C\frac{\partial}{\partial x}[F_{M_1}^0(\rho)M_1^0(\mu)M_1^0(\nu)] \tag{16}$$

which may be verified by differentiating the solution for the velocity potential of a transverse flow about an ellipsoid.

The nonlifting solution [Eq. (16)] is singular along the edge of the elliptic disk and is symmetric with respect to the y axis. Therefore, in order to satisfy the Kutta condition [Eq. (6)], we need to superimpose solutions that are singular on the edge, but asymmetric with respect to the y axis —namely, of the form $(\partial/\partial x)$ $[F_N(\rho)N(\mu)N(\nu)]$. The assumption of flow symmetry with respect to the x axis implies that only functions which are even in ν, i.e., $M_{2n+1}^m(\nu)$ and $N_{2n}^m(\nu)$, must be employed. Hence, we may postulate the following superposition for the acceleration potential:

$$\psi(\rho,\mu,\nu;\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{2n+1}^m + A_1^0(\tau)\frac{\partial}{\partial x}\psi_1^0 + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} A_{2n}^m(\tau)\frac{\partial}{\partial x}\psi_{2n}^m \tag{17}$$

where

$$\psi_{2n+1}^m = F_{M_{2n+1}}^m(\rho)M_{2n+1}^m(\mu)M_{2n+1}^m(\nu)$$
$$\psi_{2n}^m = F_{N_{2n}}^m(\rho)N_{2n}^m(\mu)N_{2n}^m(\nu) \tag{18}$$

Here F_M and F_N denote the Lamé functions of the second kind corresponding to the functions M and N, respectively.

(For complete expressions of the various Lamé functions, see Ref. 9.) Using the orthogonality properties of the ellipsoidal harmonics, we obtain the following expressions for the time-dependent lift and moment:

$$L(\tau)=\int_S [p(x,y,0^-,\tau)-p(x,y,0^+,\tau)]\,\mathrm{d}S$$

$$=-2\rho U(\tau)\int_S \psi(x,y,0^+,\tau)\mathrm{d}S$$

$$=-\frac{4}{3}\pi(1-h^2)^{-1/2}\rho U(\tau)C_1^0(\tau) \tag{19}$$

$$M(\tau)=-2\rho U(\tau)\int_S \psi(x,y,0^+,\tau)x\mathrm{d}S$$

$$=-\frac{4}{3}\pi(1-h^2)^{-1/2}\rho U(\tau)A_1^0(\tau) \tag{20}$$

In order to satisfy the Kutta condition, we have to evaluate ψ along the edge of the disk, $\rho=\mu=1$. This evaluation is identical with the steady analysis presented in Hauptman and Miloh,[1] with the only exception that here the coefficients A are functions of τ. Hence,

$$\psi_c=\psi(1,1,\nu,\tau)=\psi(x_c,y_c,0,\tau)$$

$$=x_c[1-(1-h^2)y^2-x_c^2]^{-1/2}$$

$$\times\left[A_1^0(\tau)+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}A_{2n}^m(\tau)x_c g_{2n}^m(x_c^2,y_c^2,h)\right] \tag{21}$$

as implied by Eq. (52) in Ref. 1, where

$$g_{2n}^m=\bar{N}_{2n}^m(\rho=1)\bar{N}_{2n}^m(\mu)\bar{N}_{2n}^m(\nu) \tag{22}$$

The Kutta condition requires that ψ_c vanish at the trailing edge $x_c=[1-(1-h^2)y_c^2]^{1/2}=(h^2-\nu^2)^{1/2}/h=\cos\varphi$ and it follows from Eq. (21) that

$$A_1^0+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}A_{2n}^m[\bar{N}_{2n}^m(1)]^2\cos\varphi\bar{N}_{2n}^m(h\sin\varphi)=0$$

$$-\pi/2\le\varphi\le\pi/2 \tag{23}$$

whereas at the leading edge ($\pi/2<\varphi<3\pi/2$), Eq. (21) yields the correct square-root singularity. It has been also shown in Hauptman and Miloh[1] that Eq. (23) yields general relationships between the coefficients A_{2n}^m and A_1^0, which may be written as

$$A_{2n}^m=a_{2n}^m A_1^0 \tag{24}$$

where a_{2n}^m are known coefficients that depend only on the indices n and m and on the eccentricity h. The most important coefficient is

$$a_1^0=-J(h)/I(h) \tag{25}$$

where

$$I(h)=\int_0^{\pi/2}\cos^2\varphi(1-h^2\sin^2\varphi)^{1/2}\mathrm{d}\varphi \tag{26}$$

$$J(h)=\int_0^{\pi/2}\cos\varphi(1-h^2\sin^2\varphi)^{1/2}\mathrm{d}\varphi$$

$$=\tfrac{1}{2}[(1-h^2)^{1/2}+(\text{arc}\sinh)/h] \tag{27}$$

In the sequel all the coefficients A_1^0, C_{2n+1}^m will be determined from the kinematic condition [Eq. (14)]. However, it is useful to introduce first the main results of the steady-state analysis.

Steady-State Solution

The formulation of the steady problem may be recovered from Eqs. (1-14) by letting $\partial/\partial\tau=0$ or $s=0$. The detailed analysis may be found in Refs. 1 and 10 [where the acceleration potential has been defined in nondimensional form, $\psi=(p-p_\infty)/\rho U^2$]. Here we reproduce only the most important results.

The downwash on the elliptic disk is related to ψ by

$$-\frac{\partial}{\partial x}w_s^*(x,y)=\frac{\partial\psi_s}{\partial z}(x,y,0) \tag{28}$$

or

$$w_s^*(x,y)=-\int_{-\infty}^{x}\frac{\partial\psi_s}{\partial z}\mathrm{d}x'$$

$$=-\sum_{n=0}^{\infty}\sum_{m=0}^{n}C_{2n+1}^m M_{2n+1}^m(h)Q_{2n+1}(\eta)$$

$$+A_1^0E(h)+G(x,y) \tag{29}$$

where the subscript s denotes the steady case, Q the Legendre function of the second kind, $E(h)$ the complete integral of the second kind as

$$E(h)=\int_0^{\pi/2}(1-h^2\sin^2\varphi)^{1/2}\mathrm{d}\varphi$$

and $\eta=(1-h^2)^{1/2}y$. $G(x,y)$ is a function that contains neither constant terms nor terms that are functions of y alone (for details, see Ref. 1). Hence, the first two terms on the right-hand side of Eq. (29) may represent the downwash associated with pure twist or angle of attack and the third term corresponds to the effect of camber. A given downwash w^* determines the function $G(x,y)$ and the values of the coefficients A and C. For an uncambered wing [$G(x,y)=0$] with a constant angle of attack α, the following relationships are obtained for the most important coefficients:

$$C_1^0=\frac{3J(h)}{E(h)}A_1^0=-3\left[2(1-h^2)^{1/2}+\frac{E^2(h)}{J(h)}\right]^{-1}\alpha \tag{30}$$

For a wing with a parabolic camber, we have $z^*=\zeta x^2$, $G(x,y)=w^*=2U\zeta x$, and $\partial w^*/\partial x=2U\zeta$. Thus, the following relationships are obtained:

$$C_1^0E(h)-3J(h)A_1^0=\frac{\partial w^*}{\partial x}=2U\zeta \tag{31}$$

$$\tfrac{2}{3}(1-h^2)^{1/2}C_1^0+E(h)A_1^0=0 \tag{32}$$

whereas all the other coefficients may be written as

$$C_{2n+1}^m=b_{2n+1}^m A_1^0 \tag{33}$$

Hence, the lift and moment are obtained by substituting the above values of the coefficients C_1^0 and A_1^0 in Eqs. (19) and (20).

For the uncambered wing at angle of attack, the results are

$$\frac{C_{L_s}}{\alpha}=4\left[\frac{(1-h^2)^{1/2}+E^2(h)}{(1-h^2)^{1/2}+(\text{arc}\sinh)/h}\right]^{-1},\qquad A\ge\frac{4}{\pi}$$

$$\frac{C_{L_s}}{\alpha}=4(1-h^2)^{1/2}\left[1+E^2(h)\Big/\left(1+\frac{1-h^2}{h}\ln\frac{1+h}{(1-h^2)^{1/2}}\right)\right]^{-1},$$

$$A\le\frac{4}{\pi} \tag{34}$$

$$\frac{C_{M_S}}{\alpha} = -\frac{4}{3}E(h)\left\{E^2(h) + (1-h^2)^{\frac{1}{2}}\left[(1-h^2)^{\frac{1}{2}} + \frac{(\text{arc } \sinh)}{h}\right]\right\}^{-1}, \qquad A \geq \frac{4}{\pi}$$

$$\frac{C_{M_S}}{\alpha} = -\frac{4}{3}E(h)\left[\frac{E^2(h)+1}{(1-h^2)^{\frac{1}{2}}} + \frac{(1-h^2)^{\frac{1}{2}}}{h}\ell n\frac{1+h}{(1-h^2)^{\frac{1}{2}}}\right]^{-1}, \qquad A \leq \frac{4}{\pi} \tag{35}$$

where A is the aspect ratio. It is shown in Ref. 1 that the above expressions reduce to the known extreme values as follows for the two-dimensional wing:

$$C_{L_s}/\alpha = 2\pi, \quad C_{M_s}/\alpha = -\pi/2$$

and for the slender wing:

$$\frac{C_{L_S}}{\alpha} = 2(1-h^2)^{\frac{1}{2}} = \frac{\pi}{2}A, \quad \frac{C_{M_S}}{\alpha} = -\frac{2}{3}(1-h^2)^{\frac{1}{2}} = -\frac{\pi}{6}A$$

Thus, the above analysis leads to closed-form expressions for the aerodynamic coefficients, valid in the whole range of aspect ratios between zero and infinity. This is in contrast with the classical anlaysis of Krienes,[8] which requires inversion of a different infinite set of linear equations for each aspect ratio.

The Unsteady Solution

We shall now treat the case of an uncambered wing with a constant angle of attack, moving with variable forward velocity $U(\tau)$ but without lateral motion. Hence $z^* = -\alpha x$, $w^* = -U(\tau)\alpha$, $\hat{w}^* = -\hat{U}(s)\alpha$, and Eq. (11) becomes

$$s\hat{U}(s)\alpha = \frac{\partial\hat{\psi}}{\partial z}(x,y,0^+,s) \tag{36}$$

This condition is analogous to Eq. (28) with $\partial w_s^*/\partial x = -s\hat{U}(s)\alpha$, i.e., to the case of a steady motion of a parabolically cambered wing, for which we may use Eq. (31). Hence,

$$\hat{C}_1^0(s)E(h) - 3J(h)\hat{A}_1^0(s) = -s\hat{U}(s)\alpha \tag{37}$$

Equations (33) and (24) imply that the only unknown coefficients in Eq. (17) are $\hat{C}_1^0$ and $\hat{A}_1^0$, which are related by Eq. (37). Therefore, we need an additional equation to determine these coefficients, which will be obtained from Eq. (14). Since the correct variation of the downwash with respect to x has been satisfied by Eq. (37), it is sufficient to satisfy Eq. (14) at an arbitrary point on the disk's surface. For convenience, we choose a point on the leading edge $x_1 = -(1-\eta^2)^{\frac{1}{2}}$, where $\eta^2 = (1-h^2)y^2$, $-1 \leq y \leq 1$. Hence, following Eq. (14), we get

$$\hat{U}(s)\alpha = e^{-sx_1}\int_{-\infty}^{x_1} e^{sx}\frac{\partial\hat{\psi}}{\partial z}(x,y,0,s)\mathrm{d}x \tag{38}$$

Integration by parts of the x derivative terms in Eq. (17) and substitution in Eq. (38) yield

$$\hat{U}(s)\alpha = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\hat{C}_{2n+1}^m I_{2n+1}^m + \hat{A}_1^0\frac{\partial}{\partial z}\psi_1^0 + \sum_{n-1}^{\infty}\sum_{m=0}^{n-1}\hat{A}_{2n}^m\frac{\partial}{\partial z}\psi_{2n}^m - s\hat{A}_1^0 I_1^0 - s\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\hat{A}_{2n}^m I_{2n}^m \tag{39}$$

where the functions $\partial\psi_k^m/\partial z$ are evaluated on the leading edge and

$$I_k^m = I_k^m(s,y) = e^{sx_1}\int_{-\infty}^{x_1} e^{sx}\frac{\partial}{\partial z}\psi_k^m(x,y,0)\mathrm{d}x \tag{40}$$

In the sequel, we shall develop asymptotic results for short time for various wing motions. In order to do so, we need to expand the integrals I_k^m for $s\to\infty$, which may be done by employing the Laplace integral method.[11] We provide here only the final result, i.e.,

$$I_k^m = \mu_k^m s^{-1/2} + \sigma_k^m s^{-3/2} + \ldots \tag{41}$$

where μ_k^m and σ_k^m are some known functions of y. For odd k, it may also be shown that

$$\mu_k^m = \sqrt{\frac{\pi}{2}}F_{M_k}^m(1)\bar{M}_k^m(1)\bar{M}_k^m(h\eta) \times (1-h^2)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{4}}(1-h^2\eta^2)^{\frac{1}{2}} \tag{42}$$

$$F_{M_1}^0(1) = (1-h^2)^{-\frac{1}{2}}, \quad \bar{M}_1^0 = 1 \tag{43}$$

$$\mu_1^0 = \sqrt{\frac{\pi}{2}}(1-\eta^2)^{-\frac{1}{4}}(1-h^2\eta^2)^{\frac{1}{2}} \tag{44}$$

Substituting Eq. (41) in Eq. (39), making use of Eqs. (37), (33) and (24), and keeping terms up to $s^{-3/2}$ yield

$$\hat{U}(s)\alpha\left[1 + \frac{1}{E(h)}(s^{\frac{1}{2}}\mu_1^0 + \sigma_1^0 s^{-\frac{1}{2}})\right] = \hat{A}_1^0(k_1 s^{\frac{1}{2}} + k_2 + k_3 s^{-\frac{1}{2}}) \tag{45}$$

where

$$k_1 = -\left(\mu_1^0 + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}a_{2n}^m\mu_{2n}^m\right) \tag{46}$$

$$k_2 = \frac{\partial}{\partial z}\left(\psi_1^0 + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}a_{2n}^m\psi_{2n}^m\right), \quad z = 0^+, \quad x = x_1^+ \tag{47}$$

and k_3 is a known coefficient that does not affect the leading term of the final expansion. Applying the Kutta condition [Eq. (23)] to Eqs. (46) and (47) yields the following simple expressions

$$k_1 = -2\mu_1^0 \tag{48}$$

$$k_2 = -2E(h) \tag{49}$$

Hence, from Eq. (45) we get

$$\hat{A}_1^0 = -\frac{\hat{U}(s)\alpha}{2\mu_1^0}\frac{[\mu_1^0/E(h) + s^{-\frac{1}{2}} + (\sigma_1^0/E(h)s^{-1}]}{\{1 + [E(h)/\mu_1^0]s^{-\frac{1}{2}} - (k_3/2\mu_1^0)s^{-1}\}} \tag{50}$$

Expanding the denominator of Eq. (50) yields

$$\hat{A}_1^0 = -\frac{\hat{U}(s)\alpha}{2\mu_1^0 s}\left[\frac{\mu_1^0}{E(h)} + s^{-\frac{1}{2}} + \frac{\sigma_1^0}{E(h)}s^{-1}\right]\left\{1 - \frac{E(h)}{\mu_1^0}s^{-\frac{1}{2}} + \left[\frac{k_3}{2\mu_1^0} + \frac{E^2(h)}{(\mu_1^0)^2}\right]s^{-1}\right\} \tag{51}$$

Substituting Eq. (51) in Eq. (37) yields the corresponding value of $\hat{C}_1^0(s)$; hence, the lift and moment at $\tau\to 0$ may be easily obtained by an inverse transform of the coefficients for a given $U(\tau)$.

For example, we consider below the two important cases of an impulsive acceleration to a constant speed and of a constant acceleration.

Impulsive Acceleration to Constant Speed

In this case, $U(\tau)=UH(\tau)$, where $H(\tau)$ is the unit-step Heaviside function. Hence, $\hat{U}(s)=U/s$ and Eqs. (51) and (37) reduce for $s\to\infty$ to

$$\hat{A}_1^0=-\frac{U\alpha}{E(h)}s^{-1}+\mathcal{O}(s^{-2}) \tag{52}$$

$$\hat{C}_1^0=-\frac{3J(h)}{E^2(h)}U\alpha s^{-1}+\mathcal{O}(s^{-2})-\frac{U\alpha}{E(h)} \tag{53}$$

Inverse transforms of Eqs. (51) and (52) yield, for $\tau\to0$,

$$A_1^0(\tau)=-\frac{U\alpha}{2E(h)}H(\tau)+\mathcal{O}(\tau) \tag{54}$$

$$C_1^0(\tau)=-\frac{3J(h)}{2E^2(h)}U\alpha H(\tau)+\mathcal{O}(\tau)-\frac{U\alpha}{E(h)}\delta(\tau) \tag{55}$$

where $\delta(\tau)=\mathrm{d}H(\tau)/\mathrm{d}\tau$ is the Dirac delta function.

The lift and moment may be readily obtained by substituting Eqs. (54) and (55) in Eqs. (19) and (20).

The last term in Eq. (55) gives rise to infinite lift acting only at $\tau=0$, where for $\tau=0^+$ the lift is finite. This term, associated with the impulsive acceleration, exists only in incompressible flow. In real flow, this term is replaced by a finite initial lift that is proportional to the speed of sound (see Ref. 12).

The moment and the finite starting lift at $\tau=0^+$ are determined by the leading terms in Eq. (54) and (55). Hence, the lift and moment slope (based on the maximum chord) coefficients at $\tau=0^+$ are

$$\frac{C_{L0}}{\alpha}=\frac{L(0^+)}{(\pi/2)(1-h^2)^{-1/2}\rho U^2\alpha}=\frac{4J(h)}{E^2(h)} \tag{56}$$

$$C_{M_0}/\alpha=2/(3E(h)) \tag{57}$$

where $J(h)$ is given by Eq. (27). Equations (56) and (57) are valid for $0\le h\le1$, that is, for the aspect ratio range between the circular wing and the two-dimensional airfoil. For the limit of the two-dimensional airfoil $h=1$, $E(1)=1$, and $J(1)=1/(4\pi)$, and Eq. (56) reduces to $C_{L0}/\alpha=\pi$, namely, half of the steady value. This well-known result has been first derived by Wagner.[2]

Dividing Eqs. (56) and (57) by Eqs. (34) and (35), respectively, we obtain the ratios between the starting and steady values of the lift and moment coefficients as

$$\frac{C_{L0}}{C_{L_s}}=\frac{C_{M0}}{C_{M_s}}=\frac{1}{2}+\frac{\sqrt{1-h^2}J(h)}{E^2(h)}\quad\text{for } A\ge\frac{4}{\pi} \tag{58}$$

For the range between the circular and slender wings the ellipse is transformed in such a way that the major axis $(1-h^2)^{-1/2}$ is parallel to the x axis. For this case, it may be shown that the corresponding expression to Eq. (58) is

$$\frac{C_{L_0}}{C_{L_s}}=\frac{C_{M_0}}{C_{M_s}}=\frac{1}{2}+\frac{J^*(h)}{E^2(h)},\quad\text{for } A\le\frac{4}{\pi} \tag{59}$$

where

$$J^*(h)=\int_0^{\pi/2}\cos\varphi(1-h^2\cos^2\varphi)^{1/2}\mathrm{d}\varphi$$
$$=\frac{1}{2}\left(1+\frac{1-h^2}{2h}\ell n\frac{1+h}{1-h}\right) \tag{60}$$

In the limit of slender wing ($A\to0$), $h=1$, $E(1)=1$, $J^*(1)=1/2$, and Eq. (60) reduces to $C_{L0}/C_{L_s}=1$.

For the intermediary case of the circular wing, $h=0$, $E(0)=\pi/2$, $J(0)=1$, and Eq. (58) or (59) yields $C_{L0}/C_{L_s}=1/2+4/\pi^2=0.9053$.

Constant Acceleration

For a constant acceleration $\dot{U}$, we have $U(\tau)=(2\dot{U}\tau)^{1/2}$ and, therefore, $\hat{U}(s)=(\pi\dot{U}/2)^{1/2}s^{-3/2}$. Hence, from Eqs. (49) and (37), we get

$$\hat{A}_1^0=\frac{1}{2E(h)}\left(\frac{\pi\dot{U}}{2}\right)^{1/2}\alpha s^{-3/2},\quad s\to\infty \tag{61}$$

$$\hat{C}_1^0=\frac{1}{E(h)}\left(\frac{\pi\dot{U}}{2}\right)^{1/2}\alpha\left[s^{-1/2}+\frac{3J(h)}{2E(h)}s^{-3/2}\right],\quad s\to\infty \tag{62}$$

Inverse transforms of Eqs. (61) and (62) yield for $\tau\to0$,

$$A_1^0(\tau)=\frac{1}{E(h)}\left(\frac{\dot{U}\tau}{2}\right)^{1/2}\alpha \tag{63}$$

$$C_1^0(\tau)=\frac{1}{E(h)}\left(\frac{\pi\dot{U}}{2}\right)^{1/2}\alpha\left[(\pi\tau)^{-1/2}+\frac{3J(h)}{E(h)}\left(\frac{\tau}{\pi}\right)^{1/2}\right] \tag{64}$$

Hence, the initial lift and moment due to a constant acceleration turn out to be, by Eqs. (19) and (20),

$$L_0=\frac{4}{3}\frac{\pi}{E(h)}(1-h^2)^{-1/2}\rho\dot{U}\alpha\left[1+\frac{3J(h)}{E(h)}\left(\frac{\tau}{\pi}\right)^{1/2}\right],\quad\tau\to0 \tag{65}$$

$$M_0=\frac{4}{3}\frac{\pi}{E(h)}(1-h^2)^{-1/2}\rho\dot{U}\tau,\qquad\tau\to0 \tag{66}$$

Equations (65) and (66) imply that the lift immediately attains a constant value that is proportional to the acceleration, whereas the moment at $\tau=0^+$ is zero.

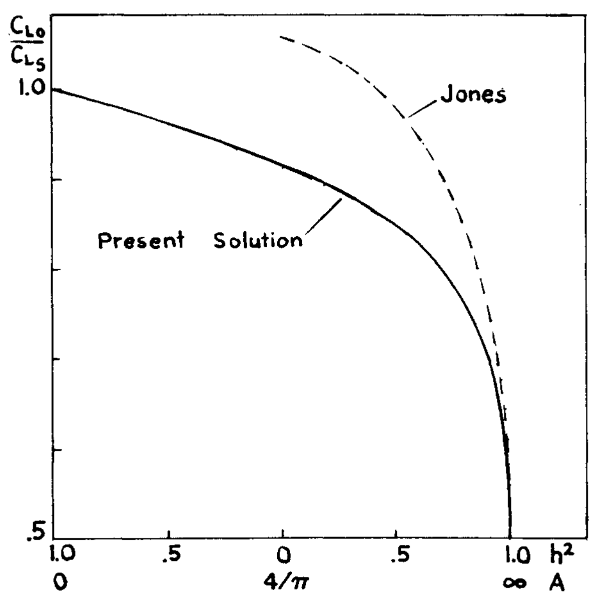

Fig. 2 Starting lift coefficient of impulsively accelerated elliptic wing.

We can now derive the ratio of the lift due to constant acceleration to the quasisteady lift for an arbitrary aspect ratio. The quasisteady lift may be written as

$$L_{qs} = \tfrac{1}{2}\rho U^2(\tau)C_{L_s}(h)\pi(1-h^2)^{-\frac{1}{2}} \tag{67}$$

where C_{L_s} is given by Eq. (34). Hence,

$$\frac{L_0}{L_{qs}} = \frac{8}{3E(h)C_{L_s}/\alpha}\frac{\dot{U}}{U^2}, \quad A \geq \frac{4}{\pi}, \quad 0 \leq h \leq 1 \tag{68}$$

$$\frac{L_0}{L_{qs}} = \frac{8(1-h^2)^{\frac{1}{2}}}{3E(h)C_{L_s}/\alpha}\frac{\dot{U}}{U^2}, \quad A \leq \frac{4}{\pi}, \quad 0 \leq h \leq 1 \tag{69}$$

The above results have been developed for an elliptic wing, meaning that the local chord varies with the span no matter how large is the aspect ratio. Therefore, in order to apply Eq. (68), in which the maximum semichord is taken as a unit length, to the limit of a two-dimensional wing with constant chord C, we must multiply Eq. (68) by the ratio $C/\bar{C}$, where $\bar{C}$ is the mean chord of the ellipse defined by

$$\bar{C} = \frac{1}{S}\int_s C(y)\mathrm{d}y = \frac{16}{3\pi} \tag{70}$$

and $C(y)$ is the local chord. Hence, for $h=1$, $E(h)=1$, $C_{L_s}=2\pi$, we obtain from Eq. (68) the well-known value for the wing of infinite span, i.e., $L_0/L_{qs} = \tfrac{1}{4}C\dot{U}/U^2$.

For the slender wing we get, putting $h=1$ and $C_{L_s}/\alpha = 2(1-h^2)^{\frac{1}{2}}$ in Eq. (69), that $L_0/L_{qs} = (4/3)(\dot{U}/U^2)$ if the maximum semichord is taken as a unit length. Therefore, for a maximum chord C, the ratio is

$$L_0/L_{qs} = \tfrac{2}{3}(C\dot{U}/U^2) \tag{71}$$

which also agrees with previous solutions for the slender wing, as shown in the next section. For the intermediary case of the circular wing, we have $h=0$, $E(0) - \pi/2$ $C_{L_s}/\alpha = 32/(8+\pi^2)$ and, therefore,

$$\frac{L_0}{L_{qs}} = \frac{8+\pi^2}{12\pi}\frac{C\dot{U}}{U^2}$$

Discussion

It is interesting to compare the present results with previous approximations. For the impulsively started elliptic wing, Jones[4] suggested the following expression for the starting lift slope coefficient:

$$C_{L_0}/\alpha = \pi/E(h) \tag{72}$$

which is exact in the limit $h \to 1$ and is valid only for large aspect ratios. Equation (72) is graphically compared with the exact solution [Eqs. (58) and (59)] in Fig. 2, where the ratio of the starting and steady lift coefficients is plotted vs the square of the eccentricity h. The result—that for a slender wing the starting lift tends to the value of steady lift as $h \to 1$ ($A \to 0$)—is consistent with the observation that the loading on a slender wing does not depend on the history of its motion.[5]

For the case of constant acceleration, we may compare Eq. (71) with the results presented by Ando[5] for the slender wing limit. Ando derived the expression $L_0/L_{qs} = \tfrac{1}{3}(C\dot{U}/U^2)$ for the slender *delta* wing. If we substitute the expression for an elliptic contour in his equation for the sectional lift distribution [Eq. (32) in Ref. 5] and integrate over the span, we obtain a result identical with Eq. (71).

It would be of great interest to derive a full generalization of the Wagner function, namely, the growth of lift as a function of arbitrary τ. In order to do so, a complete evaluation of the integrals I_k^m [Eq. (40)] for all s is necessary. This would not be an easy task, as may be seen by examining similar but even simpler integrals that occur in the unsteady analysis of the *circular* planform.[6] One of the difficulties is that those integrals are generally functions of the spanwise coordinate y and, therefore, for a given incidence Eq. (39) should be satisfied in the mean over the wing surface. [Fortunately, the integrals I_k^m affect only the higher-order terms in Eq. (52) and, therefore, the exact starting lift is obtained without this difficulty.] Therefore, a complete solution for arbitrary τ would undoubtedly involve numerical calculations. However, this was not the intention of the present paper, in which only *closed-form* solutions have been sought. Such solutions are apparently possible for small time.

Conclusions

The present paper treats analytically the fundamental lifting surface problem of an elliptic planform by expanding the acceleration potential in a series of ellipsoidal harmonics. The classical Krienes' analysis requires inversion of an infinite system of linear equations (a different system for each aspect ratio) even in the less difficult *steady* problem, in order to calculate the lift and moment. Instead, the present analysis leads to simple closed-form solutions for the small-time and steady values of the aerodynamic coefficients. For the first time, explicit expressions have been found for the initial lift of an uncambered elliptic planform due to impulsive acceleration to a constant speed, as well as the lift due to constant acceleration. These expressions are valid in the whole range of aspect ratios between zero and infinity.

More extensive treatment of the integrals associated with the unsteady downwash function [Eq. (40)] may lead to more complete solutions for transient responses, such as a full generalization of the Wagner's lift deficiency function for a finite-span wing.

The analytic expressions for the unsteady lift and moment may be also found useful as test cases of numerical schemes for the computation of steady loading on finite-aspect-ratio wings.

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